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The Study of Time Lag on Plant Growth Under the Effect of Toxic Metal: A Mathematical Model

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ABSTRACT

In this paper, a mathematical model is proposed for analytical study of plant growth subjected to the effect of toxic metal. The associated state variables are plant biomass, concentration of nutrients and concentration of toxic metal in the soil. The assumption is that the toxic metals present in the soil interferes with nutrient availability and hence, adversely affect the plant's growth. This effect is studied by introducing the time-lag (delay) in consumption and utilisation coefficient due to less availability of nutrients in the presence of toxic metal. The inclusion of delay disturbed the stability of the interior equilibrium of the system and Hopf bifurcation occurred at the critical value of delay parameter. Further, the direction, stability and period of these bifurcating periodic solutions are also studied as well as sensitivity analysis of state variables with respect to model parameters. Numerical simulation support analytical results using MATLAB.

Keywords: Concentration of nutrients, delay, Hopf bifurcation, plant biomass, toxic metal, simulation

INTRODUCTION

Plants need carbon, hydrogen, and oxygen, water and other nutrients which come from soil for growth. Nutrients are components in food that an organism uses to survive and grow. Plant-soil interaction means the mechanism in which the plants take essential nutrients from the soil through their roots which leads to growth of plants. A low concentration of heavy metals is necessary for growth of plants, but excess of these metals adversely affect the soil quality and hence retards plant growth. Thornley (1976) is the first to apply mathematical modelling to wide range of subjects in plant physiology to predict effect of factors such as temperature, humidity, radiation input and concentration of CO_2 on process rates of respiration, photosynthesis, transpiration, fluid transport and stomatal

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responses. Lacointe (2000) reported that that models suggested by Thornley were designed for a particular plant species and under particular conditions, so they could not be applicable to a wider range of conditions. Deleo et al. (1993) gave a simple model that coupled the effect of toxic metal and soil chemistry to study the adverse effect of toxic metal on biomass of trees. Guala et al. (2010, 2013) further modified the parameters of the model given by Deleo to show the model was applicable to not only trees, but all plants in general. Misra and Kalra (2012) studied how the toxicity of heavy metals could adversely affect the growth of a plant using a two-compartment mathematical model. Rouches theorem (1960) explains distribution of roots of exponential polynomials. Ruan and Wei (2001) used Rouches theorem for the discussion of distribution of roots of exponential polynomials. Kubiaczyk and Saker (2002) studied stability and oscillations in system of non-linear delay differential equations of population dynamics. Ruan and Wei (2003) used Rouches theorem for the discussion of distribution of roots of exponential polynomials for study of stability involving delays. Naresh et al (2014) studied the effect of toxicant on plant biomass with time delays. Shukla et al (1996) studied the effect of environmentally degraded soil by rain water and wind on crop yield. Sikarwar (2012) studied the effect of time delay on the dynamics of a multi team prey predator system. Naresh et al. (2006) studied the effect of an intermediate toxic product formed by uptake of a toxicant on plant biomass. Huang et al. (2016) studied analysis for global stability of system of non-linear delay differential equations involving population growth. Zhang et al. (2009) studied the distribution of the roots of a fifth-degree exponential polynomial with applications to a delayed neural network model. The explicit formulae is derived for determining the properties of the Hopf-bifurcation at the critical value using the normal form theory and manifold reduction (Hassard et al.,1981). Bocharov and Rihan (2000) came up with adjoint and direct methods for sensitivity analysis in numerical modelling in biosciences using delay differential equations. Rihan's (2003) sensitivity analysis for dynamic systems with time-lags using adjoint equations and direct methods when the parameters appearing in the model showed they were not only constants but also variables of time. Banks, Robbins and Sutton (2012) presented theoretical foundations for traditional sensitivity and generalised sensitivity functions for a general class of nonlinear delay differential equations. Theoretical results for sensitivity are presented with respect to the delays. Ingalls, Mincheva and Russel (2017) developed a parametric sensitivity analysis for periodic solutions of delay differential equations. Kalra and Kumar (2017) studied the role of time lag in plant growth dynamics using a two-compartment mathematical model. Over the last decade, a lot of work has been done in the field of plant soil interaction under the effect of toxic metals. Till date, delay differential equations have not been prominently used in the field of soil-plant dynamics and agriculture. In view of the above, a mathematical model is proposed for the study of plant growth by introducing delay parameter in containing consumption and utilisation coefficient and complex behaviour giving rise to Hopf bifurcation.

MATHEMATICAL MODEL

The plant growth dynamics is governed by the following system of non-linear delay differential equations involving three state variables: Concentration of nutrients N in the plant, amount of plant biomass W and concentration of toxic metal M in the soil .

$$\frac{dN}{dt} = (K_N - K_{NM}M) - \alpha N(t-\tau)W - \delta_1 N \tag{1}$$

$$\frac{dW}{dt} = \beta N(t-\tau)W - \delta_2 W \tag{2}$$

$$\frac{dM}{dt} = I - \gamma NM - \delta_3 M \tag{3}$$

With initial conditions $N(0)>0$, $W(0)>0$, $M(0)>0$ for all t and $N(t - \tau)$= constant for $t \in [-\tau, 0]$.

The parameters defined are : K_N is the availability of total nutrients and (K_N-K_{NM} M) is the supply of nutrients hindered due to presence of toxic metal. α is the consumption coefficient. β is the utilisation coefficient for nutrients. γ is the depletion of M due to interaction between M and N. I is the intake of toxic metal M in the soil. δ_1, δ_2, δ_3 are natural decay rates of N, W and M respectively. Here, all the parameters $\alpha, \beta, \gamma, K_N, K_{NM}, I, \delta_1, \delta_2, \delta_3$ are taken as positive.

BOUNDEDNESS

The boundedness of solutions of the model given by (1) -(3) is given by the lemma stated below:

Lemma 1. The model has all its solution lying in the region $C = \left[(N, W, M) \in {R_+}^3 : 0 \leq N + W \leq \frac{(\delta_3 K_N - I K_{NM})}{\delta_3 \varphi}, 0 \leq M \leq \frac{I}{\delta_3}\right]$, as t→∞, for all positive initial values $\{N(0), W(0), M(0), N(t-\tau) =$ Constant for all $t \in [-\tau, 0]\} \in C \subset {R_+}^3$, where $\varphi = \min(\alpha, \beta, \delta_1, \delta_2)$.

Proof: Consider the following function: $B(t) = N(t) + W(t)$

$$\frac{dB(t)}{dt} = \frac{d}{dt}[N(t) + W(t)]$$

Using Equations (1) -(2) and $\varphi = \min(\alpha, \beta, \delta_1, \delta_2)$ assuming that $N_R(t) \approx N_R(t-\tau)$ as $t \to \infty$ $\quad \frac{dW(t)}{dt} \leq \left(K_N - \frac{IK_{NM}}{\delta_3}\right) - \varphi B(t)$.

Applying the comparison theorem, as $t \to \infty$ $\quad B(t) \leq \frac{(\delta_3 K_N - I K_{NM})}{\delta_3 \varphi}$

So, $0 \leq N(t) + W(t) \leq \frac{(\delta_3 K_N - I K_{NM})}{\delta_3 \varphi}$

From equation (3): $\frac{dM}{dt} = I - \gamma MN - \delta_3 M$

$\frac{dM}{dt} \le I - \delta_3 M$, then by usual comparison theorem, when $t \to \infty$: $M \le \frac{I}{\delta_3}$

So, $0 \le M \le \frac{I}{\delta_3}$

POSITIVITY OF SOLUTIONS

Positivity means that the system sustains. For positive solutions, one needs to show that all solution of system given by Equations. (1)– (3), where initial condition is $N(0) > 0, W(0) > 0, M(0) > 0$ for all t and $N(t - \tau)$ = constant for $t \in [-\tau, 0]$, the solution $(N(t), W(t), M(t))$ of the model stays positive for all time $t > 0$.

From equation (3): $\frac{dM}{dt} \ge -(\gamma N + \delta_3)M$ i.e. $\frac{dM}{dt} \ge -\left(\frac{\gamma(\delta_3 K_N - IK_{NM}) + \delta_3{}^2\varphi}{\delta_3\varphi}\right)$

$M \ge c_1 e^{-\left(\frac{\gamma(\delta_3 K_N - IK_{NM}) + \delta_3{}^2\varphi}{\delta_3\varphi}\right)t}$, here c_1 is constant of integration. So, $M > 0$ for all t.

Similar argument holds for N and W.

INTERIOR EQUILIBRIUM OF MODEL

We calculate an interior equilibrium E^* of model. The system of equations (1) -(3) has one feasible equilibrium $E^*(N^*, W^*, M^*)$ where $N^* = \frac{\delta_2}{\beta}$,, $W^* = \left(\frac{K_N(\beta\delta_3 + \gamma\delta_2) - K_{NM}\beta I}{(\beta\delta_3 + \gamma\delta_2)}\right) - \frac{\delta_1}{\alpha}$, $M^* = \frac{\beta I}{(\beta\delta_3 + \gamma\delta_2)}$.

Theorem 1. Consider the exponential polynomial:

$$f(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) = \lambda^n + P_1{}^0\lambda^{n-1} + \cdots + P_{n-1}{}^0\lambda^n + P_n{}^0 + [P_1{}^1\lambda^{n-1} + \cdots + P_{n-1}{}^1\lambda^n + P_n{}^1]e^{-\lambda\tau_1} + \cdots + [P_1{}^m\lambda^{n-1} + \cdots + P_{n-1}{}^m\lambda^n + P_n{}^m]e^{-\lambda\tau_m},$$

Where $\tau_i \ge 0$ (i = 0,1,2,…,m) and $P_j{}^i$ (i = 0,1,2,…m :j = 1,2,…n) are constants. As $(\tau_1, \tau_2, \ldots, \tau_m)$ vary, the sum of the orders of the zeros of exponential polynomial $f(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

This result has been proved by Ruan and Wei [10,12] by using Rouches theorem.

STUDY OF INTERIOR EQUILIBRIUM AND LOCAL HOPF-BIFURCATION

Here, the dynamic behaviour of the interior equilibrium points $E^*(N^*, W^*, M^*)$ of the model given by (1) -(3) is analysed. The exponential characteristic equation about equilibrium E^* is given by:

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1\lambda^2 + n_2\lambda + n_3)e^{-\lambda\tau} = 0 \quad [4]$$

Where $m_1 = (\delta_1 + \delta_2 + \delta_3 + \gamma N^*), m_2 = (\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1 + \delta_1\gamma N^* + K_{NM}\gamma M^*)$,

$m_3 = (\delta_1\delta_2\delta_3 + \delta_2 K_{NM}\gamma M^*), n_1 = \alpha W^*, n_2 = \alpha W^*(\delta_1 + \delta_3 + \gamma N^*), n_3 = \delta_2\delta_3\alpha W^*$.

Clearly $m_1, m_2, m_3, n_1, n_2, n_3$ are all positive.

Clearly $\lambda = i\omega$ is a solution of equation (4) if and only if

$$(i\omega)^3 + m_1(i\omega)^2 + m_2(i\omega) + m_3 + (n_1(i\omega)^2 + n_2(i\omega) + n_3)e^{-i\omega\tau} = 0 \quad [5]$$

Separating real and imaginary parts:

$$m_3 - m_1\omega^2 + (n_3 - n_1\omega^2)\cos\omega\tau + n_2\omega\sin\omega\tau = 0 \quad [6]$$

$$m_2\omega - \omega^3 + n_2\omega\cos\omega\tau - (n_3 - n_1\omega^2)\sin\omega\tau = 0 \quad [7]$$

Which gives: $\omega^6 + ({m_1}^2 - {n_1}^2 - 2m_2)\omega^4 + ({m_2}^2 - {n_2}^2 + 2n_1n_3 - 2m_1m_3)\omega^2 + ({m_3}^2 - {n_3}^2) = 0$ [8]

Let a = $a = ({m_1}^2 - {n_1}^2 - 2m_2), b = ({m_2}^2 - {n_2}^2 + 2n_1n_3 - 2m_1m_3), c = ({m_3}^2 - {n_3}^2)$.

Let $\omega^2 = y$, then equation (8) becomes: $y^3 + ay^2 + by + c = 0.$ [9]

Claim 1. If $c<0$, Equation (9) contains at least one positive real root.

Proof. Let $h(y) = y^3 + ay^2 + by + c$

Here $h(0) = c < 0$, $\lim_{y\to\infty} h(y) = \infty$ o, $\exists$ $y_0 \in (0,\infty)$ such that $h(y_0)=0$.Proof completed.

Claim 2. If $c \geq 0$, then necessary condition for equation (9) to have positive real roots is $D=a^2-3b\geq0$.

Proof. Since $h(y)=y^3+ay^2+by+c$, therefore $h'(y)=3y^2+2ay+b$

$h'(y)=0$ implies $3y^2+2ay+b=0$ [10]

The roots of equation (10) can be expressed as $y_{1,2} = \frac{-2a\mp\sqrt{4a^2-12b}}{6} = \frac{-a\mp\sqrt{D}}{3}$ [11]

If $D<0$, then equation (10) does not have any real roots. So, the function $h(y)$ is monotone increasing function in y. It follows from $h(0)=c\geq 0$ that equation (9) has no positive real roots.

Clearly if $D\geq 0$, then $y_1 = \frac{-a+\sqrt{D}}{3}$ is local minima of $h(y)$. Thus the following claim.

Claim 3. If $c\geq 0$, then equation (9) has positive roots if and only if $y_1>0$ and $h(y_1)\leq 0$.

Proof. The sufficiency is obvious. Only necessity needs to be proved. Otherwise, assume that either $y_1\leq 0$ or $y_1>0$ and $h(y_1)>0$ .If $y_1\leq 0$, since $h(y)$ is increasing for $y\geq y_1$ and $h(0)=c\geq 0$, it follows that $h(y)$ has no positive real zeros. If $y_1>0$ and $h(y_1)>0$, since $y_2 = \frac{-a-\sqrt{D}}{3}$ is the local maxima value, it follows that $h(y_1)\leq h(y_2)$. Hence, $h(0)=c\geq 0$, As $h(y)$ does not have positive real roots.

Lemma 2. Suppose y_1 is defined by equation (11).

(I) If $c<0$, Equation (9) contains at least one positive real root.

(II) If $c\geq 0$ and $D=a^2-3b<0$, then equation (9) has no positive roots.

(III)If $c\geq 0$, then equation (9) has positive roots if and only if $y_1>0$ and $h(y_1)\leq 0$. **Proof.** Suppose that equation (9) has positive roots. Without loss of generality, assume that it has three positive roots, denoted by y_1,y_2,y_3. Then equation (8) has three positive roots, say $\omega_1 = \sqrt{y_1}\,, \omega_2 = \sqrt{y_2}\,, \omega_3 = \sqrt{y_3}$.

From (7) $\sin\omega\tau = \frac{m_2\omega-\omega^3}{d}$ Which gives $\tau = \frac{1}{\omega}\left[sin^{-1}\left(\frac{m_2\omega-\omega^3}{d}\right) + 2(j-1)\pi\right]$; $j = 1,2,3,-$

Let $\tau_k{}^{(j)} = \frac{1}{\omega_k}\left[sin^{-1}\left(\frac{m_2\omega_k-\omega_k{}^3}{d}\right) + 2(j-1)\pi\right]$; $k = 1,2,3.; j = 0,1,2,---$

Then is a pair of purely imaginary roots of equation (8)

Where $\tau = \tau_k{}^{(j)}, k = 1,2,3.; j = 0,1,2,---,$ $\lim_{j\to\infty} \tau_k{}^{(j)} = \infty, k = 1,2,3,4.$

Thus, define $\tau_0 = \tau_{k_0}{}^{(j_0)} = \min_{1\leq k\leq 3, j\geq 1}\left[\tau_k{}^{(j)}\right],\ \omega_0 = \omega_{k_0},\ y_0 = y_{k_0}$ [12]

Lemma 3. Suppose that $m_1>0,(m_3+d)>0,m_1 m_2-(m_3+d)>0$.

(I) If $c\geq 0$ and $D=a^2-3b<0$, then all the roots of equation (4) have negative real parts for all $\tau\geq 0$.

(II) If $c<0$ or $c\geq 0$, $y_1>0$ and $h(y_1)\leq 0$, then all the roots of equation (4) have negative real parts for all $\tau \in [0, \tau_0)$.

Proof. When $\tau=0$, equation (4) becomes

$$\lambda^3 + (m_1+n_1)\lambda^2 + (m_2+n_2)\lambda + (m_3+n_3) = 0 \quad [13]$$

Based Routh-Hurwitz's criteria, **(H1):** All roots of equation (5) have negative real parts if and only if $(m_3+n_3)>0,(m_1+n_1)(m_2+n_2)-(m_3+n_3)>0$.

If $c\geq0$ and $D=a^2-3b<0$, Lemma 2 (II) shows that equation (4) has no roots with zero real part for all $\tau\geq0$. When $c<0$ or $c\geq0$, $y_1>0$ and $h(y_1)\leq0$, Lemma 2 (I) and (III) imply that when $\tau\neq\tau_k^{(j)}$, k=1,2,3.;j≥1, equation (4) has no roots with zero real part and τ_0 is the minimum value of τ so that the equation(4) has purely imaginary roots. Applying theorem 1, the conclusion of the lemma is obtained.

Let $\lambda(\tau) = \psi(\tau) + i\omega(\tau)$ [14]

be the roots of equation (4) satisfying: $\psi(\tau_0) = 0,\ \omega(\tau_0) = \omega_0$

In order to guarantee that $\mp\omega_0$ are simple purely imaginary roots of equation (4), with $\tau=\tau_0$ and $\lambda(\tau)$ satisfies transversality condition, assume that $h'(y_0)\neq0$.

Lemma 4. Suppose $y_0=\omega_0^2$. If $\tau=\tau_0$, Then Sign $[\psi'(\tau_0)]$ ==Sign $[h'(y_0)]$

Proof. Putting $\lambda(\tau)$ in equation (4) and differentiating *w.r.t* τ, it follows that

$$\frac{d\lambda}{d\tau}\left[3\lambda^2 + 2m_1\lambda + m_2 + \left((n_1\lambda^2 + n_2\lambda + n_3)(-\tau) + (2n_1\lambda + n_2)\right)e^{-\lambda\tau}\right] = \lambda(n_1\lambda^2 + n_2\lambda + n_3)e^{-\lambda\tau}$$

Then $\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2+2m_1\lambda+m_2)e^{\lambda\tau}}{\lambda(n_1\lambda^2+n_2\lambda+n_3)} + \frac{(2n_1\lambda+n_2)}{\lambda(n_1\lambda^2+n_2\lambda+n_3)} - \frac{\tau}{\lambda}$

From equations (6) -(8):

$$\mu'(\tau_0) = Re\left[\frac{(3\lambda^2+2m_1\lambda+m_2)e^{\lambda\tau}}{\lambda(n_1\lambda^2+n_2\lambda+n_3)}\right] + Re\left[\frac{(2n_1\lambda+n_2)}{\lambda(n_1\lambda^2+n_2\lambda+n_3)}\right] = \frac{1}{\Delta}\left[3\omega_0{}^6 + 2a\omega_0{}^4 + b\omega_0{}^2\right]$$

Where $\Delta = [(n_3 - n_1\omega^2)^2 + (n_2\omega)^2]$. Here $\Delta>0$ and $\omega_0>0$.

It is concluded that Sign $[\psi'(\tau_0)]$=Sign $[h'(y_0)]$.This proves the lemma.

DIRECTION AND STABILITY OF HOPF-BIFURCATING SOLUTION

In the previous section, a family of periodic solutions is obtained that bifurcate from the positive steady state E^* at the critical values of τ. It is also worthwhile to determine the direction, stability and period of these bifurcating periodic solutions. In this section, an explicit formula will be derived to determining the properties of the Hopf-bifurcation at the critical value τ_j, using the normal form theory and manifold reduction due to Hassard, Kazarion and Wan (1981).

Let $u_1 = N - N^*, u_2 = W - W^*, u_3 = M - M^*$ and normalising the delay τ by time scaling $t \to \frac{t}{\tau}$, equations (1) -(3) are transformed into

$$\frac{du_1}{dt} = -\delta_1 u_1 - K_{NM} u_3 - \alpha W^* u_1(t-1) - \alpha u_1(t-1) u_2 \quad [15]$$

$$\frac{du_2}{dt} = -\delta_2 u_2 + \beta W^* u_1(t-1) + \beta u_1(t-1) u_2 \quad [16]$$

$$\frac{du_3}{dt} = -\gamma M^* u_1 - (\gamma N^* + \delta_3) u_3 - \gamma u_1 u_3 \quad [17]$$

Thus, work can be done in phase $C = C\left((-1,0), {R_+}^3\right)$. Without loss of generality, denote the critical value τ_j by τ_0. Let $\tau = \tau_0 + \mu$, then $\mu = 0$ is a Hopf-bifurcation value of the system given by equations (17) -(19). For the simplicity of notations, rewrite this system as

$$u'(t) = L_\mu(u_t) + F(\mu, u_t) \quad [18]$$

Where $u(t) = (u_1(t), u_2(t), u_3(t)\,)^T \in R^3, u_t(\theta) \in C$ is defined by $u_t(\theta) = u_t(t + \theta)$, and

$L_\mu: C \to R,\ \ F: R{\times}C \to R$ are given, respectively by

$$L_\mu \emptyset =$$

$$(\tau_0 + \mu)\begin{bmatrix} -\delta_1 & 0 & -K_{NM} \\ 0 & -\delta_2 & 0 \\ -\gamma M^* & 0 & -(\gamma N^* + \delta_3) \end{bmatrix}\begin{bmatrix} \emptyset_1(0) \\ \emptyset_2(0) \\ \emptyset_3(0) \end{bmatrix} +$$

$$(\tau_0 + \mu)\begin{bmatrix} -\alpha W^* & 0 & 0 \\ \beta W^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \emptyset_1(-1) \\ \emptyset_2(-1) \\ \emptyset_3(-1) \end{bmatrix}$$

And $F(\mu, \emptyset) = (\tau_0 + \mu)\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ respectively where $F_1 = -\alpha \emptyset_1(-1)\emptyset_2(0)$,

$F_2 = \beta \emptyset_1(-1)\emptyset_2(0), F_3 = -\gamma \emptyset_1(0)\emptyset_3(0)$

$$\emptyset(\theta) = \left(\emptyset_1(\theta), \emptyset_2(\theta), \emptyset_3(\theta)\right)^T \in C\left((-1,0), R\right).$$

Based on the Riesz representation theorem, there exist a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$,, such that $L_\mu \emptyset = \int_{-1}^{0} d\,\eta(\theta, 0)\emptyset(\theta)$ for $\emptyset \in C$.

Instead, choose

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{bmatrix} -\delta_1 & 0 & -K_{NM} \\ 0 & -\delta_2 & 0 \\ -\gamma M^* & 0 & -(\gamma N^*+\delta_3) \end{bmatrix}\delta(\theta) + (\tau_0+\mu)\begin{bmatrix} -\alpha W^* & 0 & 0 \\ \beta W^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\delta(\theta+1)$$

Here δ is the Dirac delta function. For $\emptyset \in C\left([-1,0], {R_+}^3\right)$, define

$$A(\mu)\emptyset = \begin{cases} \frac{d\emptyset(\theta)}{d\theta}, & \theta \in [-1,0) \\ \int_{-1}^{0} d\,\eta(\theta,0)\emptyset(\theta), & \theta = 0. \end{cases}$$ And $R(\mu)\emptyset = \begin{cases} 0, & \theta \in [-1,0) \\ F(\mu,\emptyset) & \theta = 0. \end{cases}$

Then the system (18) is equivalent to

$$u'(t) = A(\mu)\emptyset + R(\mu)u_t \qquad [19]$$

For $\psi \in C^1\left([-1,0], {R_+}^3\right)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1,0) \\ \int_{-1}^{0} d\,\eta^T(-t,0)\psi(-t), & s = 0. \end{cases}$$ And bilinear inner product

$$< \psi(s), \emptyset(\theta) > = \overline{\psi(0)}\emptyset(0) - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)\,d\xi \qquad [20]$$

Sine A^* and A=$A(0)$ are adjoint operators and $i\omega_0$ are eigen values of $A(0)$. Thus they are eigen values of A^*. Suppose that $q(\theta) = q(0)e^{i\omega_0\theta}$ is an eigen vector of $A(0)$ corresponding to the eigen value $i\omega_0$. Then $A(0)$= $i\omega_0\, q(\theta)$. When θ=0,

$\left[i\omega_0 I - \int_{-1}^{0} d\eta(\theta)e^{i\omega_0\theta}\right] q(0) = 0,$ which yields $q(0) = (1,\sigma_1,\rho_1)^T$ where

$\sigma_1 = \frac{(\alpha W^*+\delta_1+i\omega_0)}{K_{NM}}$ and $\rho_1 = \frac{\beta W^*(\delta_2-i\omega_0)}{{\delta_2}^2+{\omega_0}^2}$

Similarly, it can be verified that $q^*(s) = D(1,\sigma_2,\rho_2)e^{i\omega_0\tau_0 s}$ is the eigen value of A* corresponding to $-i\omega_0$, where

$\sigma_1 = \frac{(\alpha W^*+\delta_1-i\omega_0)}{K_{NM}}$ and $\rho_1 = \frac{\beta W^*(\delta_2+i\omega_0)}{{\delta_2}^2+{\omega_0}^2}$

In order to assure $<q^*(s), q(\theta)> = 1$, the value of D needs to be determined.

From equation (22), $<q^*(s), q(\theta)>$

$$= \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2})(1, \sigma_1, \rho_1)^T - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2}) e^{-i\omega_0\tau_0(\xi-\theta)} d\eta(\theta)(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0}\, d\xi$$

$$= \overline{D}\left\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} - \int_{-1}^{0}(1, \overline{\sigma_2}, \overline{\rho_2})\, \theta e^{i\omega_0\tau_0\theta}(1, \sigma_1, \rho_1)^T\right\}$$

$$= \overline{D}\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}W^*(\beta\rho_1 - \alpha\sigma_1)e^{i\omega_0\tau_0}\}$$

Hence, choose $\overline{D} = \frac{1}{(1+\sigma_1\overline{\sigma_2}+\rho_1\overline{\rho_2}+\tau_0\overline{\sigma_2}W^*(\beta\rho_1-\alpha\sigma_1)e^{i\omega_0\tau_0})}$

such that $< q^*(s), q(\theta) >= 1, < q^*(s), \overline{q(\theta)} >= 0.$

Following the algorithm given by Hassard et al., (1981) and using the same notations as there to compute the coordinates describing the centre manifold C_0 at μ=0. Let u_t be the solution of equation (18) with μ=0. Define

$$z(t) =< q^*(s), u_t(\theta) >,\ W(t, \theta) = u_t(\theta) - 2Re(z(t)q(\theta)) \qquad [21]$$

On the centre manifold C_0, $W(t, \theta) = W\left(z(t), \overline{z(t)}, \theta\right)$

Where $W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,$

z and $\overline{z}$ are local coordinates for centre manifold C_0 in the direction of q^* and $\overline{q^*}$. Note that W is real if u_t is real. Consider only real solution. For solution $u_t \in C_0$ of equation (20), since μ=0,

$$z'(t) = i\omega_0\tau_0 z + < \overline{q^*}(\theta), F\big(0, W(z, \overline{z}, \theta) + 2Re\big(z(t)q(\theta)\big)\big) >$$

$$= i\omega_0\tau_0 z + \overline{q^*}(0)\, F\big(0, W(z, \overline{z}, 0) + 2Re\big(z(t)q(\theta)\big)\big)$$

$$\equiv i\omega_0\tau_0 z + \overline{q^*}(0)F_0(z, \overline{z})$$

Rewrite this equation as

$$z'(t) = i\omega_0\tau_0 z(t) + g(z, \overline{z}) \qquad [22]$$

Where $g(z, \overline{z}) = \overline{q^*}(0)F_0(z, \overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} +$

$$g_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots \qquad (23)$$

As $u_t(\theta) = (u_{1t}, u_{2t}, u_{3t}\ ) = W(t,\theta) + z\, q(\theta) + \overline{z}\overline{q(\theta)}$

$(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0\theta}$, so

$$u_{1t}(0) = z + \overline{z} + W_{20}{}^{(1)}(0)\frac{z^2}{2} + W_{11}{}^{(1)}(0)\, z\overline{z} + W_{02}{}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{2t}(0) = \sigma_1 z + \overline{\sigma_1}\,\overline{z} + W_{20}{}^{(2)}(0)\frac{z^2}{2} + W_{11}{}^{(2)}(0)\, z\overline{z} + W_{02}{}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{3t}(0) = \rho_{1_1} z + \overline{\rho_{1_1}}\,\overline{z} + W_{20}{}^{(3)}(0)\frac{z^2}{2} + W_{11}{}^{(3)}(0)\, z\overline{z} + W_{02}{}^{(3)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{1t}(-1) = ze^{-i\omega_0\tau_0} + \overline{z}e^{i\omega_0\tau_0} + W_{20}{}^{(1)}(-1)\frac{z^2}{2} + W_{11}{}^{(1)}(-1)\, z\overline{z} +$$

$$W_{02}{}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{2t}(-1) = \sigma_1 e^{-i\omega_0\tau_0} z + \overline{\sigma_1}\, e^{i\omega_0\tau_0}\,\overline{z} + W_{20}{}^{(2)}(-1)\frac{z^2}{2} + W_{11}{}^{(2)}(-1)\, z\overline{z} +$$

$$W_{02}{}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots,$$

Thus, comparison of coefficients with equation (23) gives:

$$g_{20} = \overline{D}(1, \sigma_1, \rho_1) f_{z^2},\ \ g_{02} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}\ ) f_{\overline{z}^2},$$

$$g_{11} = \ \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}\ ) f_{z\overline{z}}, g_{21} = \ \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}\ ) f_{z^2\overline{z}}$$

In order to determine g_{21}, focus needs to be on computation of W_{20} (θ) and W_{11} (θ). From equations (19) and (21):

$$W' = u_t{}' - z'q - \overline{z}'q = \begin{cases} AW - 2Re\left[\overline{q^*}(0)F_0 q(\theta)\right], & \theta \in [-1,0) \\ AW - 2Re\left[\overline{q^*}(0)F_0 q(0)\right] + F_0, & \theta = 0 \end{cases}$$

Let $W' = AW + H(z, \overline{z}, \theta)$ [24]

Where $H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + H_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots$,, [25]

On the other hand, on C_0 near the origin $W' = W_z z' + W_{\overline{z}}\overline{z}'$

Expanding the above series and computing the coefficients, we get

$$[A - 2i\omega_0 I]W_{20}(\theta) = -H_{20}(\theta),\ \ AW_{11}(\theta) = -H_{11}(\theta)$$ [26]

By equation (22), for $\theta \in [-1,0)$,

$$H(z,\overline{z},\theta) = -\overline{q^*}(0)\overline{F_0}q(\theta) - \overline{q^*}(0)\overline{F_0}\,\overline{q}(\theta) = -gq(\theta) - \overline{g}\,\overline{q}(\theta)$$

Comparing the coefficients with (23) for $\theta \in [-1,0]$ that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\overline{q}(\theta),\ \ H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\overline{q}(\theta).$$

From equations (22), (25) and the definition of *A* we obtain

$$W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\,\overline{q}(\theta)$$

Solving for $W_{20}(\theta)$:

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{02}}}{3\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta},$$

And similarly

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{11}}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_2$$

Where E_1 and E_2 are both three dimensional vectors, and can be determined by setting θ=0 in *H*. In fact since $H(z,\overline{z},\theta) = -2Re\left[\overline{q^*}(0)F_0q(0)\right] + F_0$,

So

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\overline{q}(\theta) + F_{z^2},$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\overline{q}(\theta) + F_{z\overline{z}}$$

Where $F_0 = F_{z^2}\frac{z^2}{2} + F_{z\overline{z}}z\overline{z} + F_{\overline{z}^2}\frac{\overline{z}^2}{2} + \cdots$

Hence combining the definition of *A*,

$$\int_{-1}^{0} d\,\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0W_{20}(0) + g_{20}q(0) + \overline{g_{02}}\,\overline{q}(0) - F_{z^2} \text{ and}$$

$$\int_{-1}^{0} d\,\eta(\theta)W_{11}(\theta) = g_{11}q(0) - \overline{g_{11}}\,\overline{q}(0) - F_{z\overline{z}}$$

Notice that

$$\left[i\omega_0\tau_0I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right]q(0) = 0 \text{ and}$$

$$\left[-i\omega_0\tau_0I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right]\overline{q}(0) = 0$$

implies

$$\left[2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta} d\eta(\theta)\right] E_1 = F_{z^2} \text{ and } -\left[\int_{-1}^{0} d\eta(\theta)\right] E_2 = F_{z\bar{z}}$$

Hence,

$$\begin{bmatrix} (2i\omega_0 + \delta_1 + \alpha W^* e^{-2i\omega_0\tau_0}) & 0 & -K_{NM} \\ \beta W^* e^{-2i\omega_0\tau_0} & (2i\omega_0 + \delta_2) & 0 \\ \gamma M^* & 0 & (2i\omega_0 + \gamma N^* + \delta_3) \end{bmatrix} E_1 =$$

$$-2\begin{bmatrix} \alpha\sigma_1 e^{-i\omega_0\tau_0\theta} \\ -\beta\sigma_1 e^{-i\omega_0\tau_0\theta} \\ -\gamma\rho_1 \end{bmatrix} \text{and}$$

$$\begin{bmatrix} (\delta_1 + \alpha W^*) & 0 & -K_{NM} \\ \beta W^* & \delta_2 & 0 \\ \gamma M^* & 0 & (\gamma N^* + \delta_3) \end{bmatrix} E_2 = -2\begin{bmatrix} \alpha Re\{\sigma_1\} e^{i\omega_0\tau_0\theta} \\ -\beta Re\{\sigma_1\} e^{i\omega_0\tau_0\theta} \\ -\gamma Re\{\rho_1\} \end{bmatrix}$$

Thus g_{21} can be expressed by the parameters.

Based on the above analysis, each g_{ij} can be determined by the parameters. Thus, following quantities can be computed:

$$C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2} \quad , \quad \mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2Re\{C_1(0)\}, \qquad [27]$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0}$$

Theorem 2. The value of μ_2 determines the direction of the Hopf bifurcation: if $\mu_2>0(\mu_2<0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau>\tau_0$ ($\tau<\tau_0$). The value of β_2 determines the stability of bifurcating solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2<0$ ($\beta_2>0$). The value of T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2>0$ ($T_2<0$).

SENSITIVITY ANALYSIS OF STATE VARIABLES WITH RESPECT TO MODEL PARAMETERS

In this paper, the model includes constant parameters. The 'Direct Method' to estimate the general sensitivity coefficients is used. The direct method is based on considering all parameters as constants and then the sensitivity coefficients are estimated by solving sensitivity equations simultaneously with the original system. If all the parameters (α,β,γ) appearing in the system model (1)– (3) are taken to be constants, then sensitivity analysis, in this case, may just entail finding the partial derivatives of the solution with respect to each parameter. As an illustration if consider parameter β, then partial derivatives of the solution (N, W, M) with respect to β give rise to following set of sensitivity equations:

$$\frac{dS_1}{dt} = -\delta_1 S_1 - \alpha N(t-\tau)S_2 - K_{NM}S_3 - \alpha W S_1(t-\tau) \quad [28]$$

$$\frac{dS_2}{dt} = (\beta N(t-\tau) - \delta_2)S_2 + \beta W S_1(t-\tau) + WN(t-\tau) \quad [29]$$

$$\frac{dS_3}{dt} = -\gamma M S_1 - (\gamma N + \delta_3)S_3 \quad [30]$$

Where $S_1 = \frac{\partial N}{\partial \beta}, S_2 = \frac{\partial W}{\partial \beta}, S_3 = \frac{\partial M}{\partial \beta}$

Then, this system of sensitivity equations (28) – (30) along with the original system of equations (1) – (3) is solved to estimate the sensitivity of the state variables (N, W, M) to the parameter β. The similar procedure and argument holds for estimating the sensitivity of the state variables with respect to the parameters α and γ.

Sensitivity of Variables to Parameter *α*

As shown by Figures 1 and Figure 2, the parameter consumption coefficient α does not lead to much of variation and change in the values of state variables nutrient concentration N and concentration of toxic metal M which ultimately remain stable and tens to zero, as we decrease the values of α from α=0.9 to α=0.5. It predicts the lesser sensitivity of state variables N and M to the parameter α. However, for the same range of values of α, the state variable amount of plant biomass W undergoes under considerable change as shown by Figure 3. It shows increase in the rate of plant biomass with decrease in the delayed value of consumption coefficient. It remains stable as well.

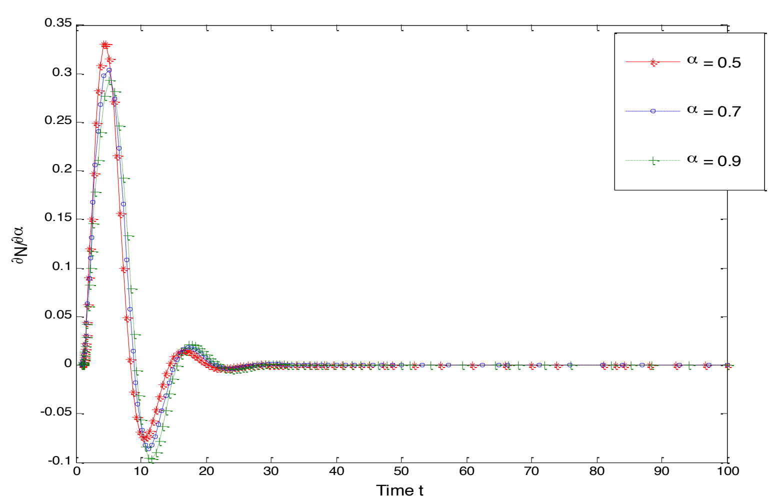

Figure 1. Time series graph between partial changes in nutrient concentration *N* for different values of consumption coefficient *α*

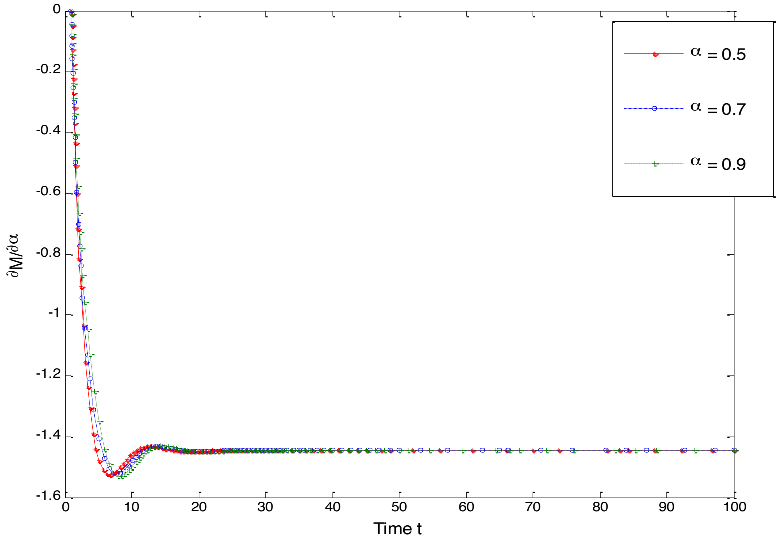

Figure 2. Time series graph between partial changes in concentration of toxic metal *M* for different values of consumption coefficient *α*

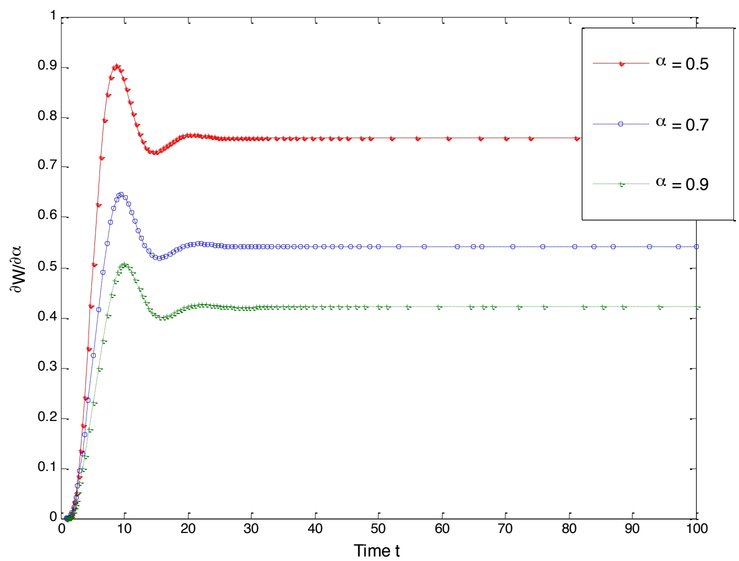

Figure 3. Time series graph between partial changes in plant biomass *W* for different values of consumption coefficient *α*

Sensitivity of Variables to Parameter *β*

Initially, the rate of nutrient concentration starts losing stability with decrease in value of utilisation coefficient, but finally becomes stable and tends to be zero as we decrease the values of parameter utilization coefficient β from β=0.7 to β=0.3 as shown by Figure 4. Figure 5 shows the increase in rate of concentration of toxic metal M with decrease in value of utilisation coefficient β from β=0.7 to β=0.3. It starts losing stability as well. Decrease in the rate of plant biomass W with decreased value of delayed utilisation coefficient is shown in Figure 6. It also starts losing stability.

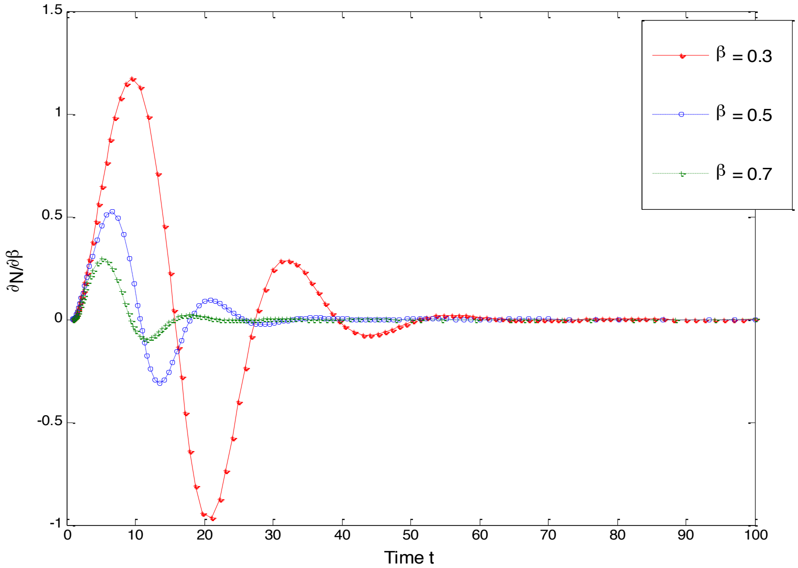

Figure 4. Time series graph between partial changes in nutrient concentration *N* for different values of utilisation coefficient β

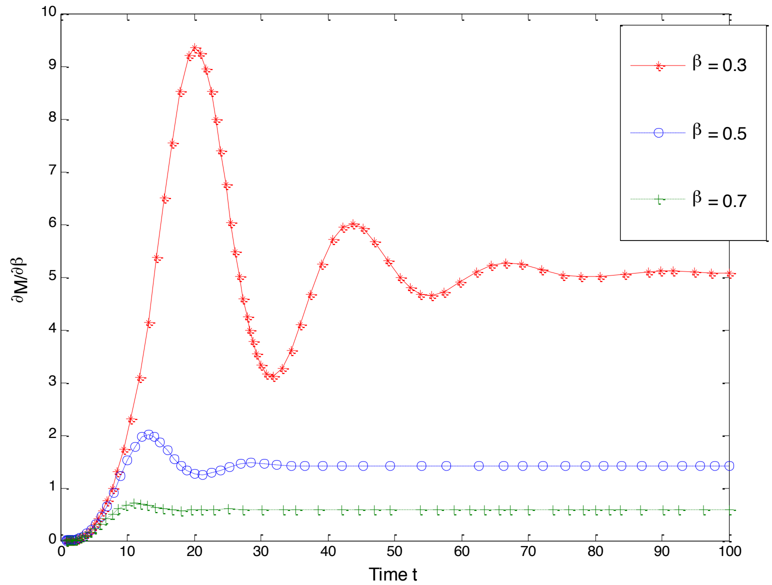

Figure 5. Time series graph between partial changes in concentration of toxic metal *M* for different values of utilisation coefficient β

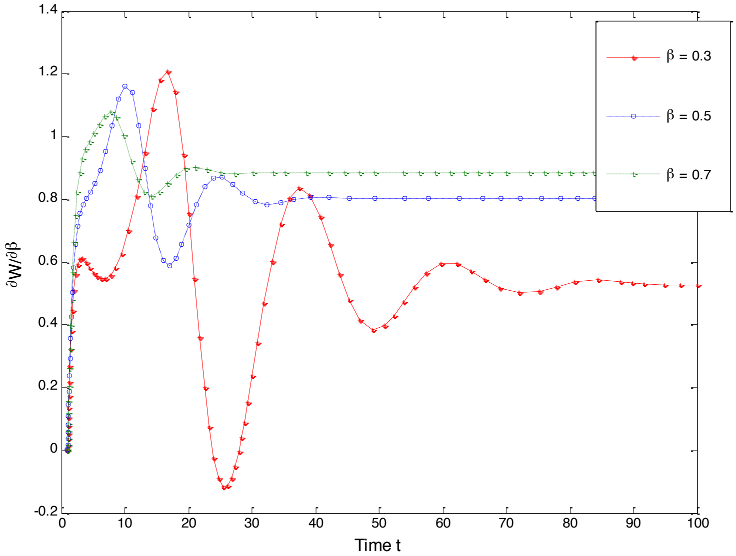

Figure 6. Time series graph between partial changes in plant biomass W for different values of utilisation coefficient β

Sensitivity of Variables to Parameter γ

Figure 7 shows nutrient concentration n is not very much affected by decease in values of depletion coefficient γ from γ=0.2 to γ=0.05. It does not lose stability. Figure 8 shows decrease in rate of concentration of toxic metal with decrease in values of depletion coefficient of toxic metal due to interaction with nutrients. It stays stable. Figure 9 shows increase in rate of plant biomass with decrease in values of depletion coefficient of toxic metal due to interaction with nutrients. It remains stable as well.

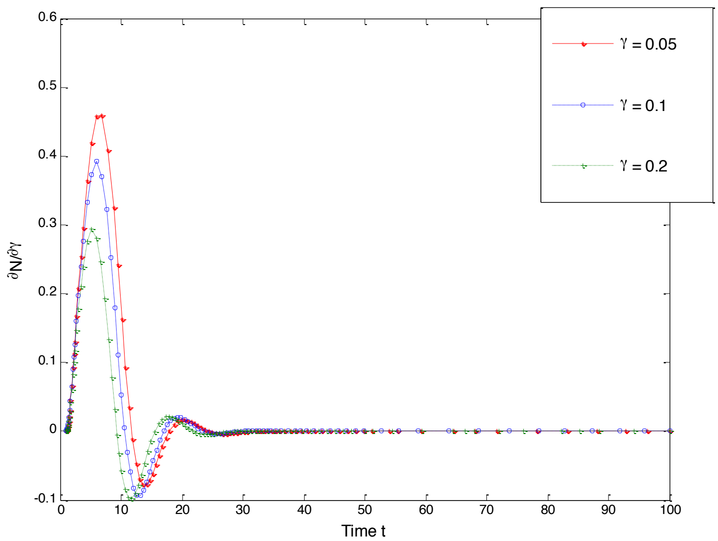

Figure 7. Time series graph between partial changes in nutrient concentration N for different values of depletion coefficient γ

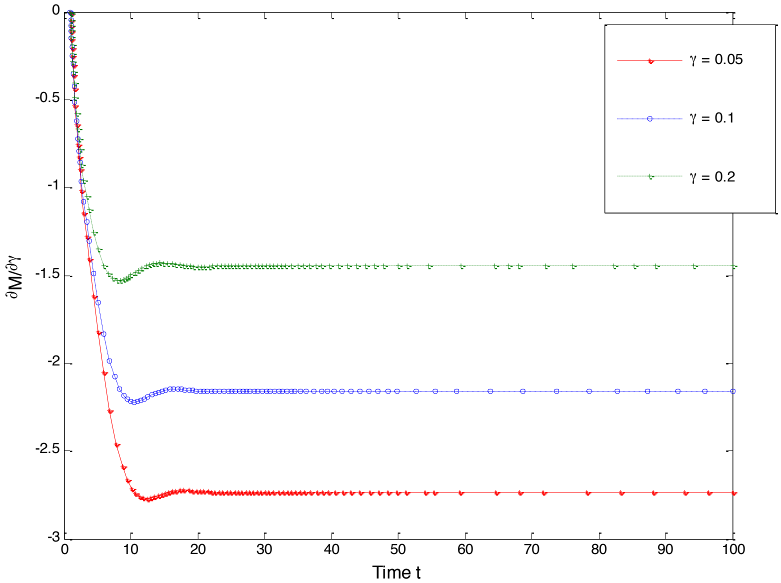

Figure 8. Time series graph between partial changes in concentration of toxic metal *M* for different values of depletion coefficient γ

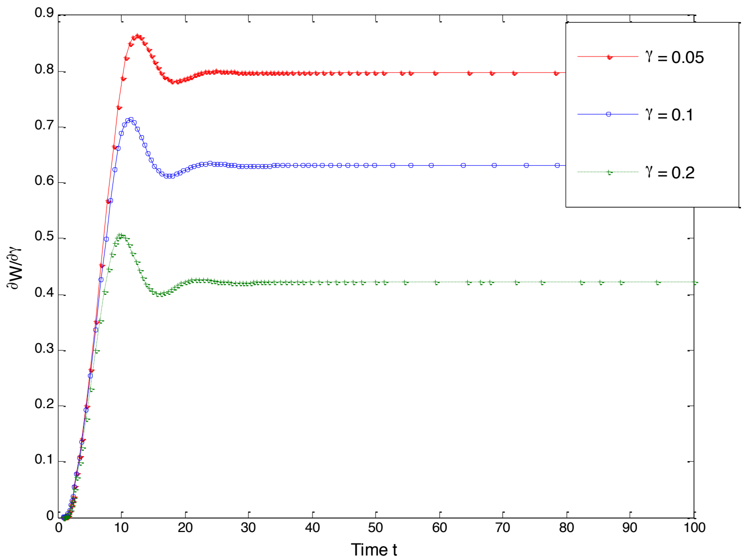

Figure 9. Time series graph between partial changes in plant biomass *W* for different values of depletion coefficient γ

NUMERICAL EXAMPLE

To consolidate the analytical result with the help of a numerical, simulation is done with MATLAB. For the following set of values, the behaviour shown by the system is as follows:

K_N=1,K_{NM}=0.3,α=0.9,δ_1=0.2,β=0.7,δ_2=0.8,I=0.5,γ=0.2,δ_3=0.4.

The behaviour of the system for different values of delay is expressed as:

E_1 (N^*=1.1426,W^*= 0.5181,M^*= 0.7950)

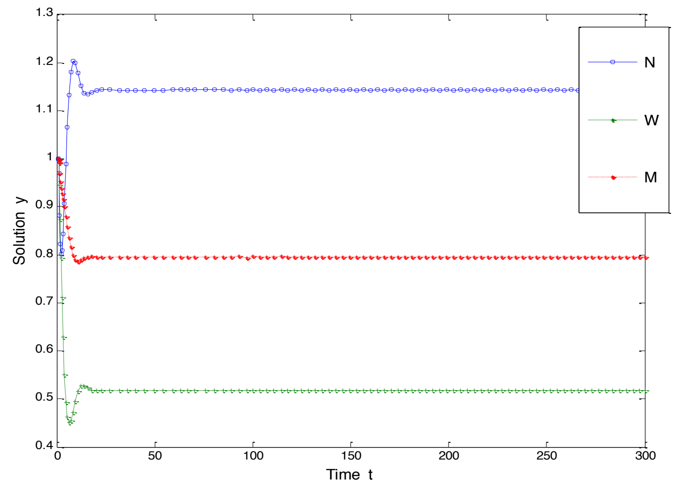

Figure 10. The interior equilibrium point E_1 (1.1426,0.5181,0.7950) of the system is stable when there is no delay that is τ=0

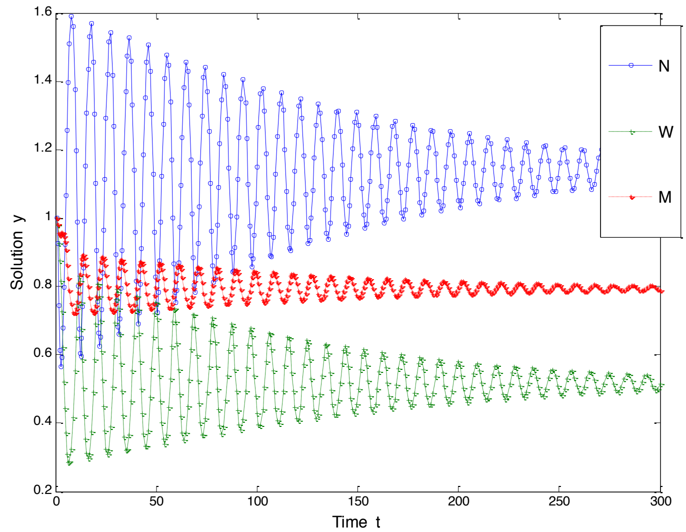

Figure 11. When delay τ<1.373, the interior equilibrium point E_1 (1.1426,0.5181,0.7950) is asymptotically stable

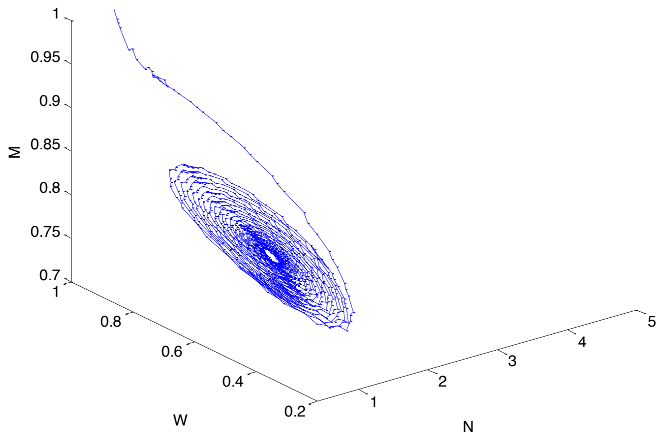

Figure 12. Phase space diagram of Nutrient N, Plant Biomass W and Toxic Metal M when delay τ<1.373

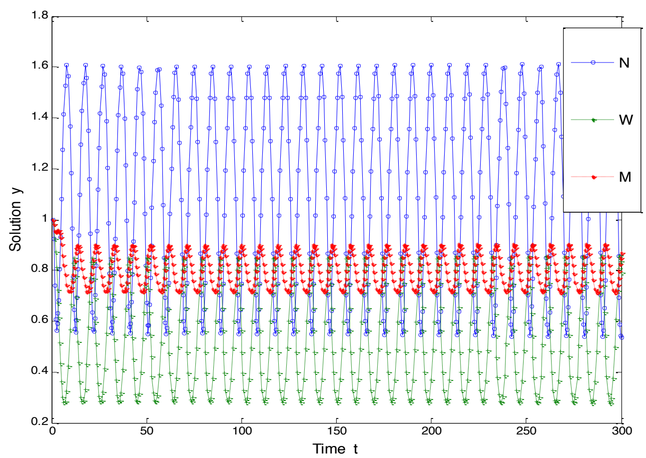

Figure 13. The interior equilibrium point E_1 (1.1426,0.5181,0.7950)losses its stability and Hopf- bifurcation occurs when delay τ≥1.373

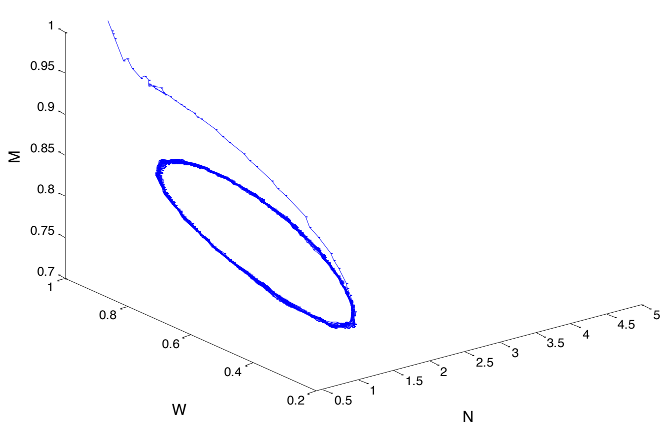

Figure 14. Phase space diagram of Nutrient N, Plant Biomass W and Toxic Metal M when delay τ≥1.373. The bifurcating periodic solution is orbitally, asymptotically stable

CONCLUSION

In this paper, a mathematical model is proposed to study the role of delay on plant growth dynamics under the effect of toxic metal. The stability and Hopf- bifurcation about the interior equilibrium is studied. It has been concluded that when there is no time delay, interior equilibrium E_1 (1.1426,0.5181,0.7950) is completely stable (Figure 10) as proved by lemma 3 using Routh-Hurwitz's criteria. But under the same set of parameters, a critical value of the parameter delay is obtained below which the system is asymptotically stable (Figure 11 and Figure 12) and unstable above that critical value of parameter (Figure 13 and Figure 14) as proved by lemma 2 and lemma 4. While passing through the critical value, the system shows oscillations that is Hopf bifurcation.

In this paper, the sensitivity of model solutions due to perturbing the parameters appearing in delay differential systems is also investigated using direct method. It is shown how the sensitivity functions enable one in identification of specific parameters and improve the understanding of the role played by specific model parameters. The oscillation and change in values accompanied by the sensitivity of state variables to parameters means that the solution is sensitive to changes in the parameter and that parameter plays an important role in the model. Sensitivity analysis reveals that the state variable nutrient concentration N is least sensitive to all parameters (α,β,γ) compared with other two state variables W and M who show considerable amount of change in their rates for different sets of values of the parameters. Rate of plant biomass shows increase with decrease in the delayed value of consumption coefficient and stays stable (Figure 3) and decreases with decrease in delayed value of utilisation coefficient and loses stability (Figure 6).

This theoretical model as well as numerical results show that for a certain threshold of parameters, the system possesses asymptotic stability around positive interior equilibrium. Further from stability analysis and numerical simulation, it is concluded that τ is a bifurcating parameter for which the interior equilibrium point shows stable oscillatory behaviour when $\tau \geq \tau_0$. After considering the effect of time lag in the system, limit cycles appear for interior equilibrium points when time delay crosses some critical value.

In future, the efforts will be made to validate the proposed mathematical model with some existing plant growth data under the effect of toxic metals. The proposed mathematical model dealing with delay in plant soil dynamics under the effect of toxicant will be helpful to farmers, agriculturists, ecologists and scientists to use pesticides, insecticides and chemical fertilizers in an optimal way. The study of the factors due to which the delay is produced, and the components being affected will help the concerned community to plan the remedial measures. Being quantitative in nature, the mathematical model will prove to be economical in terms of time and money being invested on large scale experiments.

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